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COMMUTATIVE REGULAR RINGS WITH INTEGRAL CLOSURE

BY

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ABSTRACT. First order conditions are given which are necessary for a commutative regular ring to have a prime integrally closed extension. If the ring is countable these conditions are also sufficient.

In [8] an example was given of a commutative regular ring with no prime model extension to a commutative integrally closed regular ring. In this paper we give (in §2) first order conditions which are necessary for a commutative regular ring, R, to have a prime extension to an integrally closed regular ring. In the case that R is countable these conditions are also sufficient (§3). They are, however, not sufficient in the case that R is uncountable (§4).

In [1] and [5] (see also [6] and [9]) it was shown that the theory $K^*_{\overline{CR}}$ of integrally closed commutative regular rings without minimal ($\neq 0$) idempotents is the model completion of the theory of commutative regular rings, K_{CR}. In §5 we show that a model of K_{CR} has a prime extension to a model of $K_{\overline{CR}}$ (the theory of integrally closed commutative regular rings) if and only if it has a prime extension to a model of $K^*_{\overline{CR}}$.

1. Preliminaries. A commutative ring R with unit is called regular (in the sense of von Neumann) if R satisfies $\forall x \exists y\ (x^2y = x)$. B_R is the Boolean algebra of idempotents of R. The operations of B_R are $e_1 \cup e_2 = e_1 + e_2 - e_1e_2$ and $e_1 \cap e_2 = e_1e_2$. S_R is the maximal ideal space of R. It is well known that if R is regular and commutative then S_R is the Stone space of B_R. In a natural way R is a ring of functions on S_R. ($R \ni a \longrightarrow a(\cdot)$ defined by $a(s) = a + s$ in R/s, for $s \in S_R$ any maximal ideal of R. If R is regular then R/s is a field.) It will be intuitively helpful to think of commutative regular rings as rings of functions in this way. If R is a commutative regular

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ring and e ($\neq 0$) is an idempotent of R then e is a $\{0, 1\}$ function on S_R. There is a one-to-one correspondence between clopen subsets of S_R, and idempotents of R given by $e \in R$ correspond to $\{s \in S_R | e(s) = 1\} = E_e$. We say that a relation among elements of R holds on idempotent e if for every $s \in E_e$ the relation is true mod s. We say that a relation holds at a point $s \in S_R$ if it is true mod s in R/s.

K_{CR} is the theory of commutative regular rings. $K_{\overline{CR}} = K_{CR} \cup \{$every monic polynomial has a root$\}$ is the theory of integrally closed regular rings. $K^*_{CR} = K_{CR} \cup \{$there are no minimal ($\neq 0$) idempotents$\}$ and $K^*_{\overline{CR}} = K_{\overline{CR}} \cup \{$there are no minimal ($\neq 0$) idempotents$\}$ (cf. [5]).

If T_1 and T_2, $T_1 \subset T_2$, are two theories and $\mathfrak{A} \models T_1$, $\mathfrak{A} \subset \mathfrak{L} \models T_2$ then we call $\mathfrak{L}$ a prime model extension of $\mathfrak{A}$ to a model of T_2 if whenever $\mathfrak{L}' \models T_2$ and $\phi: \mathfrak{A} \to \mathfrak{L}'$ is an embedding, there is an extension of ϕ to an embedding of $\mathfrak{L}$ into $\mathfrak{L}'$. $\mathfrak{L}$ is called minimal if $\mathfrak{A} \subset \mathfrak{L}' \subset \mathfrak{L} \models T_2$, $\mathfrak{L}' \models T_2$ implies $\mathfrak{L}' = \mathfrak{L}$. It is well known (cf. [7]) that if $\mathfrak{L}$ is a minimal prime model extension of $\mathfrak{A}$ to a model of T_2, then every prime model extension of $\mathfrak{A}$ to a model of T_2 is isomorphic with $\mathfrak{L}$.

If $R \models K_{CR}$ and $\overline{R}$ is a prime model extension of R to a model of $K_{\overline{CR}}$ then we call $\overline{R}$ an integral closure of R. It will be shown (Proposition 1) that such an $\overline{R}$ is in fact minimal and hence is the unique integral closure of R.

If $p(x) \in R[x]$, $R \models K_{CR}$, then p is called irreducible if $p(x)$ is irreducible at every point $s \in S_R$, i.e. if for every $s \in S_R$ there are no polynomials p_1 and $p_2 \in R[x]$ such that $p(x) = p_1(x)p_2(x)$ at s and degree$(p_1(x))$ and degree$(p_2(x)) > 0$ at s. $p(x)$ is called unambiguous if, at each $s \in S_R$, $p(x)$ is a power of a polynomial irreducible at s, of degree > 0 at s. It is shown in [5] that if $\phi(a_1, \ldots, a_n)$ is any quantifier free relation among $a_1, \ldots, a_n \in R \models K_{CR}$ then there is an idempotent $e \in R$ such that ϕ holds at every point of E_e and fails at every point of $S_R - E_e = E_{1-e}$. Using this is not hard to see that the Euclidean algorithm holds in R and hence that if $p(x), q(x) \in R[x]$ then $(p, q)(x) = (p(x), q(x))$, the g.c.d. of $p(x)$ and $q(x)$, is an element of $R[x]$. Using in addition the compactness of S_R it follows also that if $p(x)|q(x)$ at each point of S_R then $p(x)|q(x)$ in $R[x]$. In obtaining (p, q) from p and q it may be necessary to consider different cases of various coefficients vanishing or not on (a finite number of) different idempotents. This will often be the case in the arguments which follow. We shall not usually make explicit mention of this.

If $k \in \mathbf{N}$ let e_k be the idempotent of R on which k $(= 1 + \cdots + 1) = 0$ holds (and $k \neq 0$ holds on $1 - e_k$). Let $p(x) \in R[x]$ be unambiguous and monic of degree $n > 0$. On $1 - e_{n!}$, by considering $(p(x), p'(x)) = p_1$, $(p_1(x), p_1'(x)) = p_2(x)$ etc. one can find an irreducible polynomial $q_1(x)$ such that $q_1(x)|p(x)$ on $1 - e_{n!}$. This procedure may fail on $e_{n!}$, since at some point of $e_{n!}$ we may have $p'(x) = 0$ (or $p_1'(x) = 0$ etc.). We can, however, find a polynomial $q_2(x)$ on $e_{n!}$ such that if $s \in E_{e_{n!}}$ and $\text{char}(R/s) = k$ then, at s, $p(x) = q_2(x^{k^m})$, for some $m \in \mathbf{N}$, and $q_2(x)$ is unambiguous on $e_{n!}$. (Let e_k' be the subidempotent of e_k on which $p'(x) = 0$. On e_k' replace x^k by x in $p(x)$ to get $p_1(x)$. After a finite number of such steps one will obtain the required polynomial $q_2(x)$.)

Let $T = K_{CR} \cup \{$every monic polynomial of degree n has an unambiguous factor $|n \in \mathbf{N}\}$.

For terms used but not defined in this paper the reader is referred to [5] or [7]. Good references for the algebra in this paper are [2] and [3].

2. **Necessity.** In this section we show that if $R \models K_{CR}$ then for R to have a prime extension, $\overline{R}$, to a model of $K_{\overline{CR}}$ it is necessary that $R \models T$. We also show that if such an $\overline{R}$ exists then it is unique, up to isomorphism.

Lemma 1. *If $R, R_1 \models K_{CR}$, $R \subset R_1$ and $\alpha \in R_1$ satisfies $p(\alpha) = 0$ for some $p(x) \in R[x]$ then every maximal ideal of $R[\alpha]$ is of the form $(s \cup \{q(\alpha)\})$, $s \in S_R$ with $q(x)|p(x)$ at s and $q(x) \in R[x]$ irreducible at s.*

Proof. Let $s' \in S_R[\alpha]$. Then $s' \cap R = s \in S_R$. At s let $p(x) = p_1^{k_1}(x) \ldots p_m^{k_m}(x)$ with $(p_i, p_j) = 1$ at s. Then $p_1^{k_1}(\alpha) \ldots p_m^{k_m}(\alpha) = 0$ so, since s' is maximal and hence prime, $q(\alpha) = p_i(\alpha) \in s'$ for some i. We must show that $s \cup \{q(\alpha)\}$ generates s'. Let $a \in s'$. Then either $ae = a$ for some idempotent $e \in s$ or $a(1 - e) \in R[\alpha] - R$ for every idempotent $e \in s$. In the first case $a \in s$. In the second $a(1 - e) = r(\alpha)(1 - e)$ with $\text{degree}(r(x)) > 0$ at s. Then $r(\alpha), q(\alpha) \in s'$ so $(r, q)(\alpha) \in s'$, so $(r, q) \neq 1$ at s, i.e., $q(x)|r(x)$ at s and hence on some idempotent $1 - e, e \in s$. Hence $a \in (s \cup \{q(\alpha)\})$.

Lemma 2. *If R, α are as above then $R[\alpha] \models K_{CR}$.*

Proof. Let S' be the maximal ideal space of $R[\alpha]$ and let $q(\alpha) \in R[\alpha]$. Without loss of generality we may assume that $\text{degree}(q) < \text{degree}(p) = n$.

Let $u(x) = (p(x), q^n(x))$ and let $v(x) = p(x)/u(x)$. Then $(v(x), q(x)) = 1$ and $p(x)|v(x)q^n(x)$ so if $y \in S'$, $q(\alpha) \notin y$ if and only if $v(\alpha) \in y$. Since $(q(x), v(x)) = 1$ there exist polynomials $w_1(x), w_2(x)$ such that $q(x)w_1(x) + v(x)w_2(x) = 1$. Then $q^2(\alpha)w_1(\alpha) = q(\alpha) - q(\alpha)v(\alpha)w_2(\alpha)$. But $q(\alpha)v(\alpha) = 0$ in $R[\alpha]$ so $q^2(\alpha)w_1(\alpha) = q(\alpha)$, i.e. $R[\alpha] \models K_{CR}$.

Let ϕ: $R_1 \rightarrow R_2$ be an embedding and let $R_i \models K_{CR}$ and S_i be the

Stone space of R_i $(i = 1, 2)$. Define ϕ^*: $S_2 \to S_1$ by $\phi^*(s') = \phi^{-1}(s')$, $s' \in S_2$. Then ϕ^* is a continuous mapping of S_2 onto S_1. If $s \in S_1$ we say s splits in R_2 if $\phi^{*-1}(s)$ contains more than one point of S_2.

Lemma 3. (i) *If R, α, p are as above with p irreducible at s then s does not split in $R[\alpha]$ and if N is a neighborhood of s $(\mathrm{id}^{*-1}(s))$ in $S_{R[\alpha]}$ then there is a neighborhood M of s in S_R such that* $\mathrm{id}^{*-1}(M) \subset N$. (id: $R \to R[\alpha]$.)

(ii) *If $\overline{R}$ is a prime extension of R to a model of $K_{\overline{CR}}$ then no $s \in S_R$ splits in $\overline{R}$.*

Proof. (i) The fact that s does not split in $R[\alpha]$ follows from the irreducibility of $p(x)$ at s by Lemma 1. Let N be clopen in $S_{R[\alpha]}$. Then N is defined by some idempotent $q(\alpha) \in R[\alpha]$ by $N = \{s' \in S_{R[\alpha]} \mid q(\alpha)(s') = 0\}$. Hence $p(x) \mid q(x)$ at s and hence in some neighborhood M of s. Clearly $\mathrm{id}^{*-1}(M) \subset N$.

(ii) Let $s \in S_R$. There is an $R_1 \models K_{\overline{CR}}$ with $R \subset R_1$ such that s does not split in R_1. R_1 can be obtained by repeatedly adjoining roots of polynomials irreducible at s (more precisely $(\mathrm{id}^*)^{-1}(s)$). Since $\overline{R}$ is prime over R we have $R \xrightarrow{\mathrm{id}} \overline{R} \xrightarrow{\phi} R_1$. Since s does not split in R_1, $(\phi \circ \mathrm{id})^{*-1}(s)$ contains only one point of S_{R_1}. So $\mathrm{id}^{*-1}(s)$ contains only one point of $S_{\overline{R}}$.

Remark. If $R \subset \overline{R}$ are models of K_{CR} and no point of S_R splits in $\overline{R}$ then id^* is a homeomorphism and hence $B_R = B_{\overline{R}}$, i.e. R and $\overline{R}$ have the same idempotents.

Proposition 1. *If $R \models K_{CR}$ and $\overline{R}$ is a prime extension of R to a model of $K_{\overline{CR}}$ then $\overline{R}$ is minimal and hence unique.*

Proof. From the above remark R and $\overline{R}$ have the same idempotents. Every element of $\overline{R}$ is algebraic over R, i.e. satisfies some $p(x) = 0$ for $p(x) \in R[x]$. Suppose $R \subset R_1 \subset \overline{R}$ with $R_1 \models K_{\overline{CR}}$. Let $\alpha \in \overline{R}$ satisfy $p(\alpha) = 0$, $p(x) \in R[x]$. Let $p(x) = \Pi(x - \beta_i)$ in R_1. Then at each point of $S_{\overline{R}} = S_{R_1} = S_R$, α equals one of the β_i. Then there are idempotents $e_i \in B_{\overline{R}} = B_R$ such that $\alpha = \Sigma e_i \beta_i$. Hence $\alpha \in R_1$, i.e. $R_1 = \overline{R}$.

Lemma 4. *If $R \models K_{CR}$, $R \not\models T$ then there is a polynomial $p(x) \in R(x)$ and a point $s \in S_R$ such that $p(x)$ is irreducible at s but in every neighborhood N of s $p(x)$ is ambiguous.*

Proof. Let $p(x)$ be a monic polynomial of lowest degree not having an unambiguous factor. Let $X = \{s \in S_R \mid$ at s, $p(x)$ is ambiguous$\}$. It is clear that X is open since if $p(x) = p_1(x)p_2(x)$ with $(p_1(x), p_2(x)) = 1$ at s then this holds in some neighborhood of s. X cannot be closed. For if it were, X would be compact and equal to E_e for some idempotent e. Using the compactness of X we could factor $p(x) = q(x)r(x)$ on X $(= Ee)$ with both $q(x)$ and $r(x)$ having positive degree everywhere on X. Then at least one of $q(x)$, $r(x)$, say $q(x)$, would not have an unambiguous factor on X. There will be a subidempotent e_1 of e on which $q(x)$ is monic and does not have an unambiguous factor. Let $q(x)$ have degree m on e_1. Then $p_1(x) = q(x)e_1 + x^m(1 - e_1)$ does not have an unambiguous factor, is monic and is of lower degree than $p(x)$. This would contradict the way $p(x)$ was chosen. Consequently $\overline{X} - X \neq \emptyset$. Let $s \in \overline{X} - X$.

Theorem 1. *If* $R \models K_{CR}$ *and* $R \not\models T$ *then* R *does not have a prime extension to a model of* $K_{\overline{CR}}$.

Proof. Let R_1 be a candidate for such an extension. Then by the remark following Lemma 3, $B_{R_1} = B_R$ and $S_{R_1} = S_R$. Let s, $p(x)$ be as in Lemma 4. Since we are only interested in $p(x)$ in a neighborhood of s we can take $p(x)$ to be monic. Let $p(x) = \Pi_{i=1}^n (x - \alpha_i)$ in R_1. Let $X = \{x \in S_R | p$ is ambiguous at $s\}$. Then as in the proof of Lemma 4, X is open. Let $X_1, \ldots, X_{n-1}$ be disjoint copies of X, say $X_i = \phi_i(X)$, the ϕ_i homeomorphisms, and let $S' = S \cup X_1 \cup \cdots \cup X_{n-1}$. We define a topology on S' by specifying the basic neighborhood of points of S'. If $s \in X$, a basic neighborhood for s is any open subset of X containing s. Similarly if $s \in X_i$. If $s \in S - \overline{X}$, a basic neighborhood of s is any open subset of $S - \overline{X}$ containing s. If $s \in \overline{X} - X$ the basic neighborhoods of s are of the form $N \cup \bigcup_{i=1}^{n-1} \phi_i (N \cap X)$ where N is a neighborhood of s in S.

It is clear that S' is a totally disconnected compact Hausdorff space containing S. All we have done is glued on $n - 1$ copies of X along the boundary of X. $R[\alpha_1, \ldots, \alpha_n]$ can be considered as a ring of functions on S' as follows: If $a \in R[\alpha_1, \ldots, \alpha_n]$ then $a(s)$ is defined for $s \in S$, if $s \in X_i$ let $a(s) = a(\phi_i^{-1}(s))$. Let α' be the following function on S'.

$$\alpha' = \begin{cases} \alpha_1 & \text{on } S, \\ \alpha_{i+1} & \text{on } X_i. \end{cases}$$

Consider $R[\alpha']$. Since p is irreducible at s and $p(\alpha') = 0$, s does not split in $R[\alpha']$. Also by Lemma 2, $R[\alpha'] \models K_{CR}$. By repeatedly adjoining roots of polynomials which are irreducible at s we can find $R_2 \models K_{\overline{CR}}$ with $R_2 \supset R[\alpha']$ and s not splitting in R_2. We shall show that there is no embedding ϕ: $R_1 \rightarrow R_2$. Suppose there were. $\phi(\alpha_1), \ldots, \phi(\alpha_n)$ are roots of p in R_2

with $p(x) = \Pi_{i=1}^n (x - \phi(\alpha_i))$ in R_2. α' is a root of p in R_2 so there are idempotents e_i in R_2 such that, on e_i, $\alpha' = \phi(\alpha_i)$ and $\bigcup E_{e_i} = S_{R_2}$. Hence in some neighborhood, N of s in S_{R_2}, $\alpha' = \phi(\alpha_1)$, say. By Lemma 3 there is a neighborhood M of s in S_R such that $\phi^{*-1}(M) \subset N$. Let s_1 be a point of M at which p is ambiguous, say $p(x) = p_1(x)p_2(x)$, $(p_1(x), p_2(x)) = 1$ at s_1 and suppose $p_1(\alpha_1) = 0$ at $s_1 \in S_{R_1} = S_R$. Hence $p_1(\phi(\alpha_1)) = 0$ at each point of $\phi^{*-1}(s_1)$. But at some point of $\phi^{*-1}(s_1)$, $p_2(\alpha') = 0$ and hence $p_1(\alpha') \neq 0$. This contradiction shows that no such ϕ exists.

3. **Sufficiency in the countable case.** In this section we show that if $R \models T$, $\overline{\overline{R}} = \aleph_0$ then R has a prime extension to a model of $K_{\overline{CR}}$.

Lemma 5. *If* $R \models K_{CR}$ *and* $p(x) \in R[x]$ *is unambiguous and* $R[\alpha_i] \subset R_i \models K_{CR}$ *with* $p(\alpha_i) = 0$ *in* R_i *for* $i = 1, 2$ *then* $R[\alpha_1] \simeq R[\alpha_2]$.

Proof. Since $p(x)$ is unambiguous $S_R = S_{R[\alpha_1]} = S_{R[\alpha_2]}$ and hence $R[\alpha_1]$, $R[\alpha_2]$ are both rings of functions on S_R. The canonical mapping taking $\alpha_1 \to \alpha_2$ gives an isomorphism at each point $s \in S_R$ and hence gives an isomorphism of $R[\alpha_1]$ and $R[\alpha_2]$.

Lemma 6. *If* $R \models T$ *and* $p(x) \in R[x]$ *is unambiguous and* $p(\alpha) = 0$ *then* $R[\alpha] \models T$. $(R[\alpha] \subset R' \models K_{\overline{CR}}.)$

Proof. We must prove that if $q(x)$ is a polynomial in $R[\alpha][x]$ then q has an unambiguous factor over $R[\alpha]$. Let $q = q(\alpha, x)$ and let $q_1(x) = \Pi_{i=1}^n q(\alpha_i, x)$. (The α_i are distinct roots of p in some extension, $R'$of $R[\alpha]$ in which p splits. Since q_1 is symmetric in the α_i, q_1 is actually a polynomial in $R[x]$, since every symmetric polynomial of the α_i's is a polynomial of the coefficients of $p(x)$.) Let $q_2(x)$ be an unambiguous factor of $q_1(x)$ in $R[x]$ such that $(q_2, q) \neq 1$ at any point $s \in S$. We shall show that in $R[\alpha]$ q_2 can be factored as a product of unambiguous polynomials. This will give the desired result for q_1.

First we remark that we can take p and q_2 to be irreducible and separable. For let p_s and q_s be the irreducible separable polynomials corresponding to q and q_2 as in §1, and suppose that $p_s(\beta) = 0$ and that q_s is a product of unambiguous factors in $R[\beta]$. It is clear that q_2 is then a product of unambiguous factors in $R[\beta]$ and that these factors remain unambiguous in in $R[\alpha]$. (This follows at each point of S_R from elementary theorems about fields.)

Hence we must show that if $p(x)$, $q(x) \in R[x]$ are irreducible and $p(\alpha) = 0$ then in $R[\alpha]$, $q(x)$ is a product of unambiguous (in fact irreducible) factors.

Let $p(x)$ be monic of degree n and $q(x)$ monic of degree m. This may involve considering different cases on different idempotents. This is no problem since if $e \in R$ is an idempotent then $R = eR \oplus (1-e)R$. Let

$$T(\mathbf{t}, \mathbf{u}, x) = \prod_{\sigma \in S_n; \nu \in S_m} \left(x - \sum t_i \alpha_{\sigma(i)} - \sum u_j \beta_{\nu(j)}\right)$$

where the α_i's are distinct roots of p and the β_j's distinct roots of q and S_n is the symmetric group on n letters, S_m the symmetric group on m letters and the t_i, u_j are new variables. $T(\mathbf{t}, \mathbf{u}, x)$ is a polynomial over R since it is symmetric in the α_i and β_j. Let $\mu = m!n! + 1$, and let $T(x) = \Pi T(\mathbf{t}, \mathbf{u}, x)$ where the product is taken over all vectors $\mathbf{t}$ and $\mathbf{u}$ with entries from the set $\{1, 2, \ldots, \mu\}$, if they are distinct. Recall that e_p is the idempotent on which $p = 0$ holds. On e_p for $p < \mu$ these elements will not be distinct, so we must consider what happens on these idempotents separately. Let $e_{p,k}$ be the subidempotent of e_p on which every coefficient of $p(x)$ and $q(x)$ belongs to the finite field with p^k elements (i.e. satisfies $t^{p^k} - t = 0$). On each $e_{p,k}$ the lemma follows by considering a finite number of different cases, each of which behaves just like a finite field. On $f = e_p - \bigcup_{p^k \leq \mu} e_{p,k}$ there is a coefficient a (of $p(x)$ or $q(x)$) such that $a, a^2, \ldots, a^\mu$ are distinct at every point of f. On this idempotent we can use this set of distinct elements instead of $\{1, 2, \ldots, \mu\}$.

Let $T(x) = r(x)\Pi r_i(x)$ be an unambiguous (in fact irreducible) factoring of T with degree$(r) \geq$ degree(r_i) for all i. Let δ be a root of r. Since r is unambiguous $S_{R[\delta]} = S_R$. At each point $s \in S_R$, $p(x)$ and $q(x)$ split in $R[\delta]$. This is clear since the degree $[R[\delta]/s : R/s]$ will be the degree of the splitting field F of $p(x)$ and $q(x)$ at s over R/s, and $R[\delta]/s \subset F$. Notice that if $w(\delta)$ is a root of p (or q) at s then $w(\delta)$ is a root of p (or q) in some neighborhood of s. Since S_R is compact we can find $\alpha_i(\delta), \beta_i(\delta) \in R[\delta]$ such that in $R[\delta]$ $p(x) = \Pi_{i=1}^n (x - \alpha_i(\delta))$ and $q(x) = \Pi_{i=1}^n (x - \beta_i(\delta))$. The α_i and β_i are polynomials in δ. Consider the following sets

$$V_{ij} = \{1, \alpha_1^2, \ldots, \alpha_1^{n-1}, \beta_i, \beta_i^2, \ldots, \beta_i^j\}.$$

Each element of V_{ij} is a linear (over R) combination of $1, \delta, \delta^2, \ldots, \delta^{k-1}$ (k = degree r) which are linearly independent over R. Hence for each i, j there is an idempotent e_{ij} on which the set V_{ij} is linearly dependent over R. e_{ij} can be found as the set where suitable determinants vanish. On $e_{i,j+1} - e_{ij}$, β_i satisfies an irreducible polynomial of degree $j + 1$ over

$R[\alpha_1]$. Also $e_{i,m-1} = 1$. It is now easy to see that we can piece together irreducible factors of $q(x)$ over $R[\alpha_1]$. This proves the lemma.

Theorem 2. *If $R \models T$ and $\overline{\overline{R}} = \aleph_0$ then R has a prime extension to a model of $K_{\overline{CR}}$.*

Proof. Let $\overline{R} = R[\alpha_i \mid i < \omega]$ be such that α_{i+1} satisfies an unambiguous polynomial over $R[\alpha_1, \ldots, \alpha_i]$ and $\overline{R} \models K_{\overline{CR}}$. Since R is countable it is clear that such an $\overline{R}$ exists by Lemma 6. Lemma 5 shows that $\overline{R}$ is prime over R, since $R[\alpha_1, \ldots, \alpha_i][\alpha_{i+1}]$ is prime over $R[\alpha_1, \ldots, \alpha_i]$.

Remark. We have actually proved a little more than Theorem 2. We actually have that if $R \models T$ and $\{p_i | i \in \omega\}$ is a countable set of polynomials in $R[x]$ then there is a prime way to extend R to a model of K_{CR} in which all the polynomials $p_i(x)$ split.

Theorems 1 and 2 together give the following result of [4].

Corollary 1. *If $R \models K_{CR}$ and $\overline{\overline{R}} = \aleph_0$ then R has a prime extension to a model of $K_{\overline{CR}}$ if and only if $R \models T$.*

In this countable case the necessity of T can be proved by using the elimination of quantifiers for $K^*_{\overline{CR}}$ (in the language with an additional function symbol f defined by $\forall x(x^2 f(x) = x \wedge x f^2(x) = f(x))$, cf. [5]), and Ehrenfeucht's theorem.

4. The uncountable case. In this section we give an example of an $R \models T$, $\overline{\overline{R}} = 2^{\aleph_0}$ without a prime extension to a model of $K_{\overline{CR}}$. Let $\overline{Q}$ denote the algebraic closure of the rationals and let $\{x_i \mid i < \omega\}$ be algebraically independent. Let $F = \overline{Q}(x_1, x_2, \ldots)$ and let $\overline{F}$ be the algebraic closure of F. Let $R_1 = F^{\omega}$ and R_2 be the subring of R_1 of all sequences which are constant except on finite sets. There exist $2^{\aleph_0}$ sequences $f_\alpha\colon \omega \rightarrow \{x_i \mid i < \omega\}$, $\alpha < 2^{\aleph_0}$ such that if $\alpha \neq \beta$ $(\alpha, \beta < 2^{\aleph_0})$ then $\mathrm{range}(f_\alpha) \cap \mathrm{range}(f_\beta)$ is finite. Let $R_3 = R_2[f_\alpha \mid \alpha < 2^{\aleph_0}]$ and let R be the smallest regular subring of R_1 containing R_3. It is not hard to show that $B_R = B_{R_2}$ (= all finite or cofinite subsets of ω) and that $R \models T$. Let R_4 be the subring of $\overline{F}^{\omega}$ of all sequences which are constant except on finite sets. R_4 is the prime extension of R_2 to a model of $K_{\overline{CR}}$. Let $R_5 \subset \overline{F}^{\omega}$ be a model of $K_{\overline{CR}}$ containing R, with $S_R = S_{R_5}$ $(= \omega + 1)$. We shall construct a model $R_6 \supset R$ of $K_{\overline{CR}}$ such that there is no embedding of $R_5 \rightarrow R_6$, over R. This will show that R has no prime extension to a model of $K_{\overline{CR}}$.

Let $\{\phi_\alpha \mid \alpha < 2^{\aleph_0}\}$ be all the distinct embeddings of R_4 into R_4 over R.

Notice that if $R_4 \subset R_6$ and ϕ: $R_4 \to R_6$ is an embedding and $S_{R_6} = S_R$ then ϕ has at most one extension to an embedding $R_5 \to R_6$ since if $a \in R_5$ then $ae_i \in R_4$ for each $e_i = (\delta_{ij})_{j\in\omega}$ and $\phi(a)$ is determined by the $\phi(ae_i)$. For each $\alpha < 2^{\aleph_0}$ choose $q_\alpha \in \overline{F}^\omega$ such that $q_\alpha^2 = f_\alpha$ and $\phi_\alpha^{-1}(q_\alpha) \notin R_5$ (i.e. there is no $h \in R_5$ such that $\phi_\alpha(he_i) = q_\alpha e_i\ \forall i \in \omega$). Such a q_α exists since R_5 contains only countably many square roots of f_α. Let R_6 be an extension of $R_2[q_\alpha \mid \alpha < 2^{\aleph_0}]$ to a model of $K_{\overline{CR}}$ with $S_{R_6} = S_R$. Suppose that ϕ: $R_5 \to R_6$ were an embedding. Then $\phi|_{R_4} = \phi_\alpha$ for some α. Let $x^2 - f_\alpha = (x - h_1)(x - h_2)$ in R_5. Let $E_1 = \{s \in S_{R_6} \mid \phi(h_1)(s) = q_\alpha(s)\}$ and $E_2 = S_R - E_1$. These must be clopen subsets of $S_{R_6} = S_{R_5}$ corresponding to idempotents e_1 and e_2, say. But then $q_\alpha = \phi(h_1)e_1 + \phi(h_2)e_2 = \phi(h_1e_1 + h_2e_2)$ which is a contradiction.

The reason that R has no prime extension to a model of $K_{\overline{CR}}$ is that there is too much interweaving among elements of R. It may be possible, by excluding this kind of interweaving, to arrive at some quite general sufficient condition for $R \models T$ to have a prime extension to a model of $K_{\overline{CR}}$.

5. **Prime extensions to models of K_{CR}^* and $K_{\overline{CR}}^*$.** In this section we show that every model of K_{CR} has a prime extension to a model of K_{CR}^* and that if $R \models K_{CR}$ then R has a prime extension to a model of $K_{\overline{CR}}$ if and only if R has a prime extension to a model of $K_{\overline{CR}}^*$.

Proposition 2. *Let $R \models K_{CR}$ then R has a prime extension to a model of K_{CR}^*.*

Proof. Let X be the set of isolated points of S_R. For each $x \in X$ let C_x be a copy of the Cantor set. Let S_R^* be obtained from S_R by replacing each $x \in X$ by C_x (in the obvious way). R can be considered as a ring of functions on S_R^* (constant on each C_x). Let B_x be the set of idempotents corresponding to clopen subsets of C_x. Let $B = \bigcup_{x\in X} B_x$ and let $R^* = R[B]$. It is clear that $R \subset R^* \models K_{CR}^*$. We must show that R^* is prime over R. So let ϕ: $R \to R_1 \models K_{CR}^*$. Let e_x be the idempotent corresponding to $x \in X$. Observe that B_x is the unique countable atom free Boolean algebra. Hence for each $x \in X$ there is a copy of B_x, say B_x' with $\phi(e_x)$ as largest element (i.e. B_x' is a Boolean algebra of subidempotents of $\phi(e_x)$ with $\phi(e_x)$ as 1). Let $B' = \bigcup_{x\in X} B_x'$. Then ϕ extends naturally to an embedding of R^* onto $\phi(R)[B'] \subset R_1$.

Proposition 3. *If $R \models K_{CR}$ then R has a prime extension to a model of*

$K^*_{\overline{CR}}$ *if and only if* R *has a prime extension to a model of* $K_{\overline{CR}}$.

Proof. Proposition 2 shows that if R has a prime extension to a model of $K_{\overline{CR}}$ then R has a prime extension to a model of $K^*_{\overline{CR}}$. Conversely suppose that R_1 is a prime extension to a model of $K^*_{\overline{CR}}$. Let $X \subset S_R$ be the set of isolated points of S_R and let e_x be the idempotent corresponding to $x \in X$. Let ν: $R \to R_1$ be the embedding of R in R_1 and let $\overline{x} \in \nu^{*-1}(x)$. Observe that if $x \in S_R - X$ then x does not split in R_1 since we can construct a model R_2 of $K^*_{\overline{CR}}$ containing R in which x does not split by first constructing R^* as in Proposition 2 and then extending R^* to a model of $K_{\overline{CR}}$ in which x does not split. Define $\widetilde{R}_1$ as a ring of functions on S_R as follows: for each $a \in R_1$ let $\widetilde{a}(x) = a(\nu^{*-1}(x))$ for $x \in S_R - X$ and $\widetilde{a}(x) = a(\overline{x})$ for $x \in X$; $\widetilde{R}_1$ is the ring of functions $\widetilde{a}$ for $a \in R_1$. It is not hard to see that $R_1 \simeq \widetilde{R}^*_1$ (as constructed in Proposition 2). We omit the details. Suppose that $R \subset \overline{R} \models K_{\overline{CR}}$. There is an embedding ϕ: $R_1 \to \overline{R}^*$ since R_1 is prime over R. The correspondence $\widetilde{R}_1 \ni a \to a \; (\in \widetilde{R}^*_1 = R_1) \to \phi(a) \; (\in \overline{R}^*) \to \phi(a) \in \overline{R}$ defines an embedding of $\widetilde{R}_1$ into $\overline{R}$. Hence $\widetilde{R}_1$ is the required extension of R.

Remark. If $R \models K_{CR}$ has a prime extension to a model of K^*_{CR} (or $K^*_{\overline{CR}}$) then this prime extension is unique. If S_R is not perfect then it is not minimal.

Proposition 3 combined with Theoerms 1 and 2 shows that for $R \models K_{CR}$ to have a prime extension to a model of $K^*_{\overline{CR}}$ it is necessary that $R \models T$ and that if $\overline{\overline{R}} = \aleph_0$ then this condition is also sufficient.

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